

Inverse Problems in Space Physics

Bernd Inhester, May 2002

The program

Part I: Examples

- Image deblurring
- Tomography
- Radiative transfer inversion
- Helioseismology

Part II: Mainly direct methods

- Fourier transforms
- Singular value decomposition
- Backus-Gilbert or Mollifier method

Part III: Mainly Iterative methods

- Noise and a priori knowledge
- Iteration algorithms
- Regularization by Tikhonov
- Nonlinear problems
- Gencode and Neural networks

Literature

Introductions to Inverse Problems

I.J.D. Craig and J.C. Brown, *Inverse Problems in Astronomy*, Adam Hilger Ltd. Bristol, 1986. (An introduction, we dont have it in our library, but I have a copy)

J.A. Scales and M.L. Smith, *Introductory Geophysical Inverse Theory*, Samizdat Press, 1997. (freely available in internet: <http://samizdat.mines.edu>)

More comprehensive treatments are

A. Tarantola, *Inverse Problems*, Elsevier, 1987. (In the library at [G6 81], a bit old-fashioned)

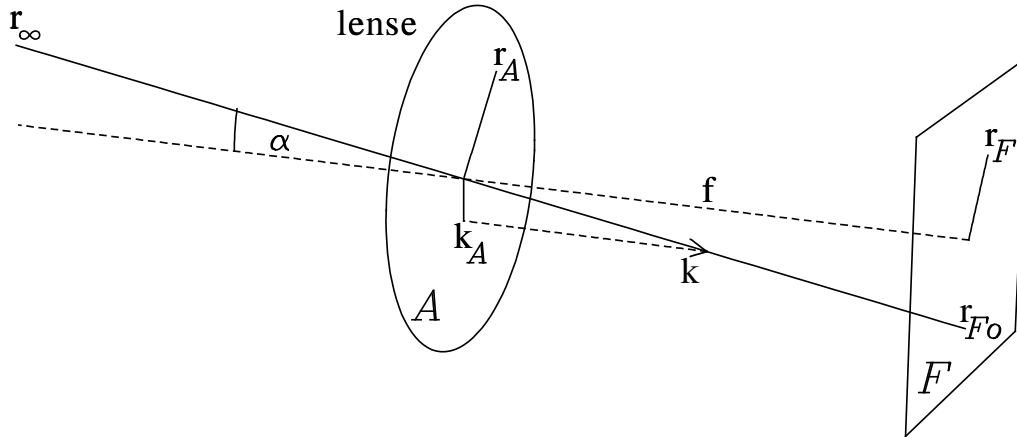
R. Parker, *Geophysical Inverse Theory*, Princeton, 1994. (Standard reference for gephysicists, often quoted, should perhaps be bought by the library)

A first aid for practical problem solving is good old

W.H. Press, B. Flannery, S.A. Teukolsky, W.T. Vetterling, *Numerical Recipies*, Cambridge University Press, 1986. (The library code is [G6 70]. There are heavily revised more recent editions, the latest version is available in the internet:

www.ulib.org/webRoot/Books/Numerical_Recipies)

Image deblurring: The phase function



Geometry for the calculation of the point spread function, \mathcal{A} is the plane of the aperture with a lens in front, \mathcal{F} is the focal plane at a distance f from the aperture. An object is assumed infinitely away in direction of \mathbf{r}_∞ and has its geometrical image at \mathbf{r}_{F_0} in the focal plane.

Electrical wave field in focal plane F from a point source at \mathbf{r}_∞ is

$$\mathbf{E}(\mathbf{r}_F, t) = \mathbf{E}_0 \frac{k e^{-i\omega t}}{i2\pi f} \int_A e^{i\Phi(\mathbf{r}_F, \mathbf{r}_A)} d^2(\mathbf{r}_A)$$

There are three contributions to the final phase of the field on the focal plane

$$\Phi(\mathbf{r}_F, \mathbf{r}_A) = \underbrace{\mathbf{k}_A \cdot \mathbf{r}_A}_{\substack{\text{init phase} \\ \text{in front of} \\ \text{lens}}} + \underbrace{\Delta\phi_{\text{lense}}(r_A)}_{\substack{\text{phase change due} \\ \text{to lense}}} + \underbrace{k|\mathbf{r}_F + \mathbf{f} - \mathbf{r}_A|}_{\substack{\text{phase change due to pro-} \\ \text{pagation from plane } \mathcal{A} \\ \text{to plane } \mathcal{F}}}$$

Making rigorous use of the Fraunhofer approximation

$$f \gg r_A \gg r_F$$

and
$$\Delta\phi_{\text{lense}} \equiv \phi_0 - \sqrt{f^2 + r_A^2}, \quad \mathbf{r}_{F_0} \equiv \frac{f}{k} \mathbf{k}_A$$

gives
$$\Phi(\mathbf{r}_F, \mathbf{r}_A) = \phi_0 - \frac{k}{f} (\mathbf{r}_F - \mathbf{r}_{F_0}) \cdot \mathbf{r}_A$$

Image deblurring: The point spread function

For a circular aperture area A with radius R we have

$$\begin{aligned} \int_A e^{i\Phi(\mathbf{r}_F, \mathbf{r}_A)} d^2(\mathbf{r}_A) &= e^{i\phi_0} \int_A e^{-i\frac{k}{f}(\mathbf{r}_F - \mathbf{r}_{F_0}) \cdot \mathbf{r}_A} d^2(\mathbf{r}_A) \\ &= 2\pi R^2 \frac{J_1\left(\frac{kR}{f}|\mathbf{r}_F - \mathbf{r}_{F_0}|\right)}{\frac{kR}{f}|\mathbf{r}_F - \mathbf{r}_{F_0}|} e^{i\phi_0} \end{aligned}$$

which is essentially the 2D Fourier transform of the aperture area A where the wave number is $|\mathbf{r}_F - \mathbf{r}_{F_0}|/f$ times the wave number k of the electromagnetic wave

→ final field is

$$\mathbf{E}(\mathbf{r}_F, t) = \mathbf{E}_0 \frac{kR^2}{f} \frac{J_1\left(\frac{kR}{f}|\mathbf{r}_F - \mathbf{r}_{F_0}|\right)}{\frac{kR}{f}|\mathbf{r}_F - \mathbf{r}_{F_0}|} e^{i(\phi_0 - \frac{\pi}{2} - \omega t)}$$

→ intensity in the focal plane

$$\begin{aligned} I(\mathbf{r}_F) &= \frac{c}{2} |\mathbf{E}(\mathbf{r}_F, t)|^2 = \\ &= \underbrace{\pi R^2 \frac{c}{2} \mathbf{E}_0^2}_{P_0} \frac{1}{\pi} \left(\frac{kR}{f}\right)^2 \left(\frac{J_1\left(\frac{kR}{f}|\mathbf{r}_F - \mathbf{r}_{F_0}|\right)}{\frac{kR}{f}|\mathbf{r}_F - \mathbf{r}_{F_0}|}\right)^2 \end{aligned}$$

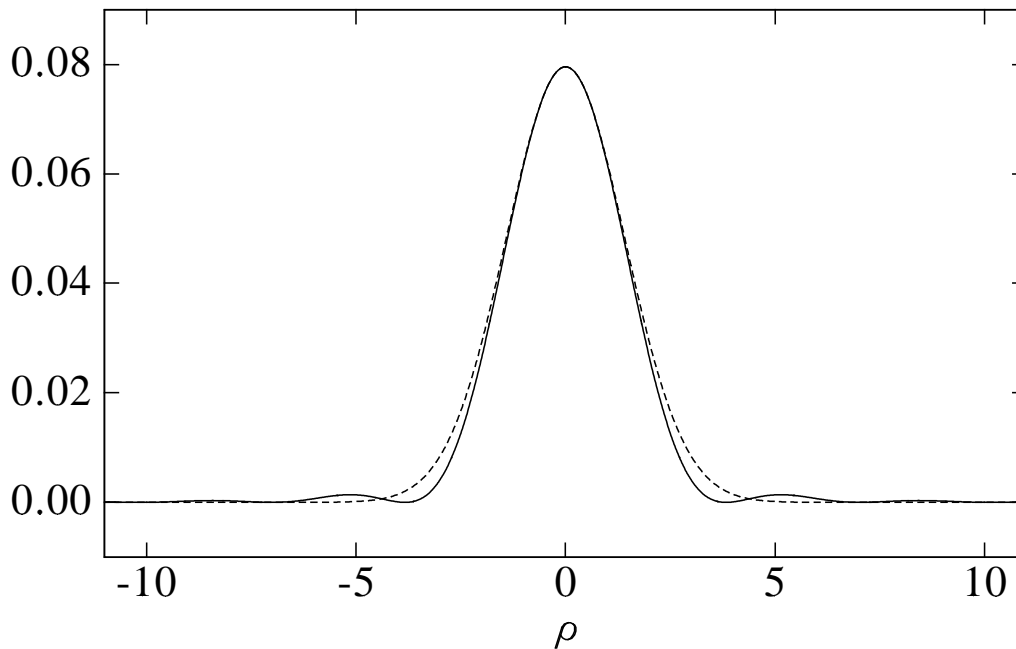
P_0 (power collected in aperture area)

If instead of a discrete point source ideally focussed at \mathbf{r}_{F_0} , we have a distributed source of brightness, we finally have to replace

$$P_0 \longrightarrow I_0(\mathbf{r}_{F_0}) d^2\mathbf{r}_{F_0}$$

Image deblurring: The inverse problem

Point spread function in diffraction limit



Cross section through the point spread function $(1/\pi)(J_1(\rho)/\rho)^2$ and a Gaussian $(1/4\pi) \exp -(\rho/2)^2$ of similar shape (dashed).

The final expression is

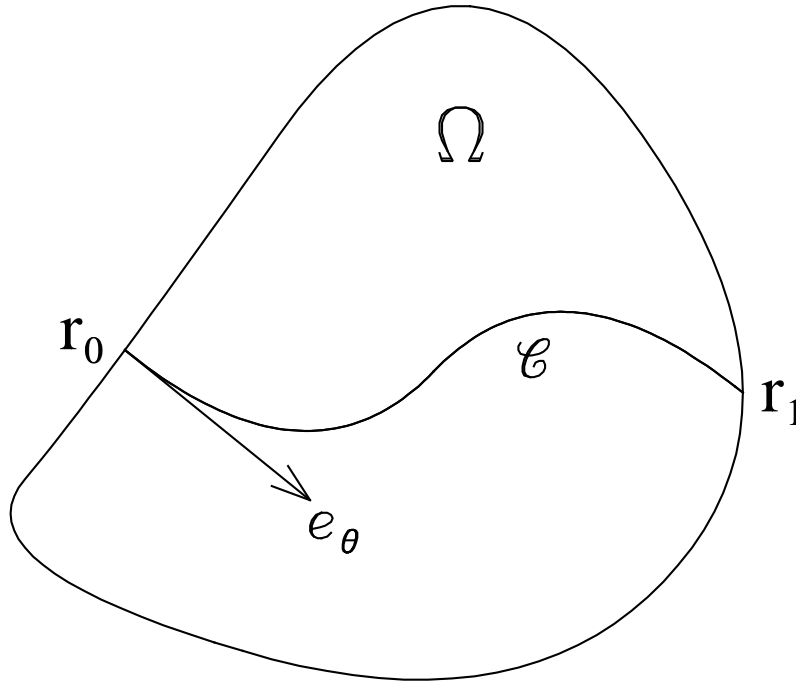
$$\underbrace{I(\mathbf{r}_F)}_{\text{Data}} = \int_F \underbrace{\frac{1}{\pi} \left(\frac{kR}{f}\right)^2 \left(\frac{J_1\left(\frac{kR}{f}|\mathbf{r}_F - \mathbf{r}_{F_0}\right)}{\frac{kR}{f}|\mathbf{r}_F - \mathbf{r}_{F_0}|}\right)^2}_{\text{Kernel}} \underbrace{I_0(\mathbf{r}_{F_0})}_{\text{Model}} d^2\mathbf{r}_{F_0}$$

- If we know I_0 we can calculate what intensity I we would observe (Forward problem – straight forward integration).
- Usually we observe I and would like to know what the original distribution I_0 looks like (Inverse problem – much harder to solve).

Image deblurring: A practical example

Tomography: General ray paths

Tomography aims to derive the distribution of a parameter in the interior of a domain Ω from observations of a diagnostic wave field on the boundary $\partial\Omega$.



Diagnostic ray path \mathcal{C} through domain Ω from \mathbf{r}_0 to \mathbf{r}_1

If, e.g., the refractive index is known, the ray path \mathcal{C} from any \mathbf{r}_0 to \mathbf{r}_1 can be calculated and the attenuated intensity

$$I_1(\mathbf{r}_0, \mathbf{r}_1) = I_0 \exp \left[- \int_{\mathcal{C}(\mathbf{r}_0, \mathbf{r}_1) \cap \Omega} \kappa(\mathbf{r}) d\mathbf{r} \right]$$

can be measured. Deducing the local absorption $\kappa(\mathbf{r})$ from these measurements is an inverse problem – a hard one depending on the ray paths \mathcal{C} .

Tomography: The X-ray transform

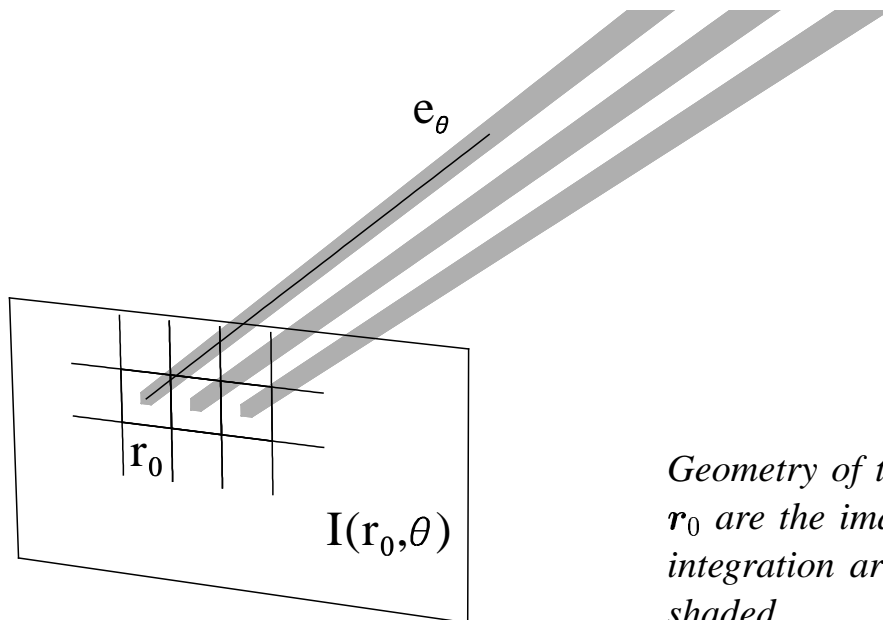
Choose the diagnostic wave so that refractive index is close to unity and κ moderate. So rays are straight along their initial direction \mathbf{e}_θ

$$\mathcal{C} = \{\mathbf{r}_0 + s\mathbf{e}_\theta \mid s \in \mathbb{R}\}$$

Let $s = s_0$ and s_1 be the intersections of \mathcal{C} with $\partial\Omega$, then

$$\underbrace{\ln\left(\frac{I_0 - I_1(\mathbf{r}_0, \theta)}{I_0}\right)}_{\text{data}} = \int_{s_0}^{s_1} \kappa(\mathbf{r}_0 + s\mathbf{e}_\theta) ds = \int_{\Omega} K_\theta(\mathbf{r}_0) \underbrace{\kappa(\mathbf{r})}_{\text{model}} d^3\mathbf{r}$$

kernel: $K_\theta(\mathbf{r}_0) = \int_{s_0}^{s_1} \delta(\mathbf{r}_0 + s\mathbf{e}_\theta - \mathbf{r}) ds$



Geometry of the X-ray transform. \mathbf{r}_0 are the image pixel center, the integration areas (rays) are grey-shaded.

- To investigate a 3D body, take a 2D manifold of positions \mathbf{r}_0 (image) with a 1D manifold of directions (the scan directions θ should cover $[0, \pi]$)
- If the \mathbf{e}_θ all lie in a plane the 3D X-ray transform decomposes into a set of 2D transforms which can all be solved independently.

Tomography: The Radon transform

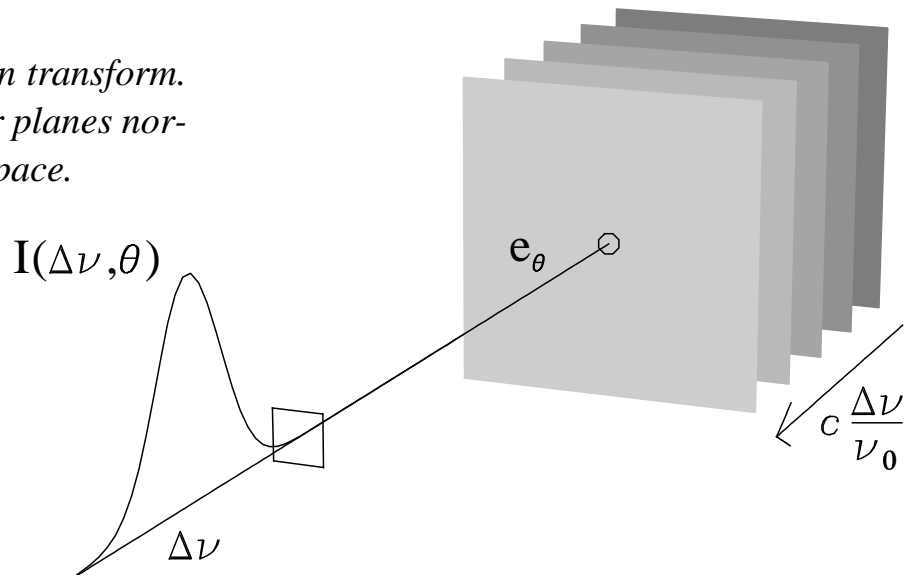
We are measuring the line emission from an optically thin plasma cloud with distribution function $f(\mathbf{v})$ from different directions \mathbf{e}_θ .

The intensity I at frequency ν offset from the line center by $\Delta\nu = \nu - \nu_0$ is proportional to the number of particles which have a velocity component $v = c\Delta\nu/\nu_0$ in direction \mathbf{e}_θ .

$$\underbrace{I(\Delta\nu, \theta)}_{\text{data}} = \frac{\int I d\nu}{N} \int_{\mathbf{v} \cdot \mathbf{e}_\theta = c \frac{\Delta\nu}{\nu_0}} \underbrace{f(\mathbf{v})}_{\text{model}} d^3\mathbf{v}$$

where $N = \int f d^3\mathbf{v}$ (total number of emitting particles) and $\int I d\nu$ is independent of direction \mathbf{e}_θ it is measured in, c is the speed of light.

*Geometry of the Radon transform.
The integration is over planes normal to \mathbf{e}_θ in velocity space.*



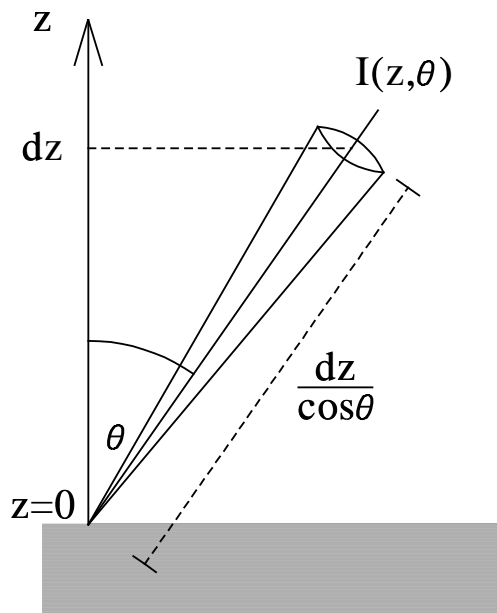
- To get the full 3D distribution measure a 1D manifold of Doppler-shifts $\Delta\nu$ (spectra) in a 2D manifold of directions (the scan directions θ should cover a half sphere).
- In 2D, the X-ray and the Radon transform are actually identical except that \mathbf{e}_θ is rotated by $\pi/2$.

Radiative transfer: The transport equation

In many atmospheric problems the radiance I_ν at a frequency ν depends only on height z and propagation angle θ .

The radiance $I(z, \theta)$ propagating at an angle θ with respect to the vertical \hat{z} is modified locally by absorption and thermal emission

$$\cos \theta \frac{d}{dz} I_\nu(z, \theta) = - \underbrace{\kappa_\nu(z) I_\nu(z, \theta)}_{\text{absorption}} + \underbrace{\epsilon_\nu}_{\text{thermal emission}} \quad (\text{no scattering})$$



Geometry for the derivation of the radiative transport equation.

Another common approximation is local thermodynamic equilibrium

$$\epsilon_\nu = \kappa_\nu(z) B_\nu(T(z)) \quad \text{where}$$
$$B_\nu(T(z)) = \frac{2h\nu^3}{c^2} \frac{1}{\exp\left(\frac{h\nu}{k_B T(z)}\right) - 1}$$

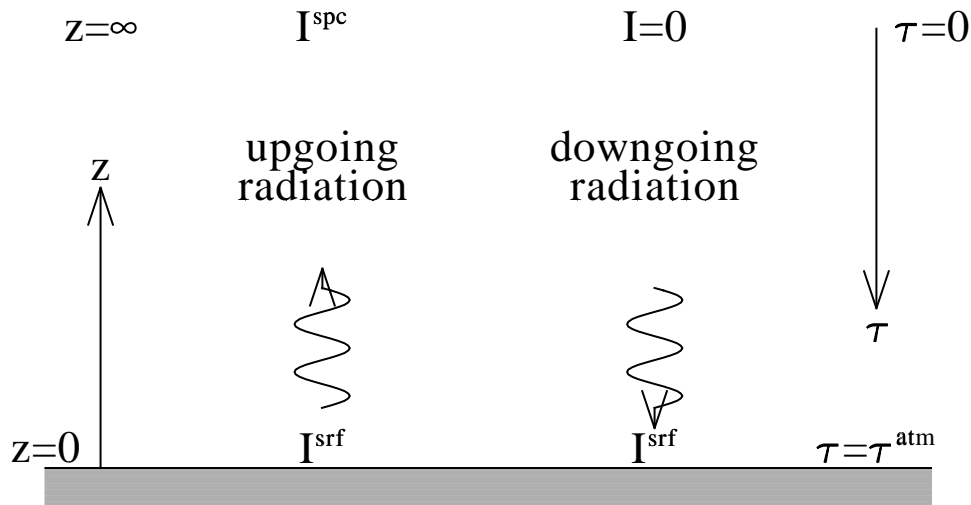
is Planck's function at the local temperature.

Radiative transfer: Up/downward radiances

The integration is simplified by introducing the frequency dependent optical depth

$$\tau_\nu(z) = \int_z^\infty \kappa_\nu(z') dz' \quad \text{hence} \quad \kappa_\nu dz = -d\tau_\nu$$

The optical thickness of the entire atmosphere is $\tau_\nu(z=0) \equiv \tau_\nu^{\text{atm}}$.



Up/downward radiance boundary values.

Integration for upward propagation ($\cos \theta > 0$) gives the radiance we may observe in space

$$I_\nu^{\text{spc}} = I_\nu^{\text{srf}} e^{-\tau_\nu^{\text{atm}} / \cos \theta} + \int_0^{\tau_\nu^{\text{atm}}} B_\nu(T(\tau_\nu)) e^{-\tau_\nu / \cos \theta} \frac{d\tau_\nu}{\cos \theta}$$

Integration for downgoing radiation ($\cos \theta < 0$) yields the radiance we observe on the ground when looking upwards

$$I_\nu^{\text{srf}} = \int_0^{\tau_\nu^{\text{atm}}} B_\nu(T(\tau_\nu)) e^{-(\tau_\nu^{\text{atm}} - \tau_\nu) / |\cos \theta|} \frac{d\tau_\nu}{|\cos \theta|}$$

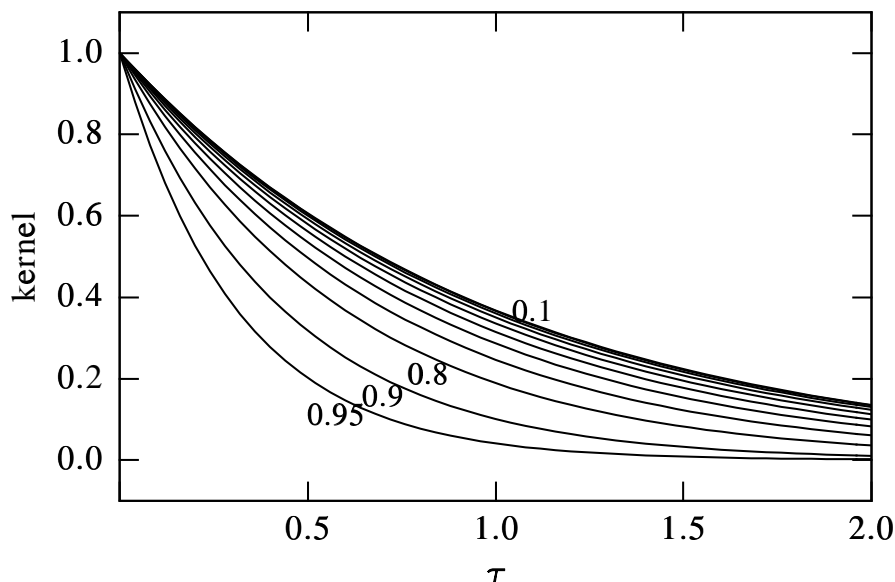
Radiative transfer: Solar limb darkening equation

The radiance from the Sun for optical ν depends on θ , or, equivalently, on the relative distance $\sqrt{1 - \cos^2 \theta}$ from the Sun's center

$$\underbrace{I_\nu(\cos \theta)}_{\text{data}} = \frac{1}{\cos \theta} \int_0^\infty \underbrace{e^{-\tau_\nu / \cos \theta}}_{\text{kernel}} \underbrace{B_\nu(T(\tau_\nu))}_{\text{model}} d\tau_\nu$$

This equation can be used to infer $B_\nu(T(\tau_\nu))$, hence $T(\tau_\nu)$ from measurement of $I_\nu(\cos \theta)$.

- limb darkening \Rightarrow increase of T with τ_ν
- Kernel $\exp -\tau_\nu / \cos \theta$ is smooth and sensitive only where it varies with $\cos \theta$, i.e., for $\tau_\nu \simeq 0.1 \dots 1.5$ (lower chromosphere).



Kernel of the limb darkening equation for $\rho = \sqrt{1 - \cos^2 \theta} = 0.1, 0.2, \dots, 0.8, 0.9, 0.95$. The solution of the inversion problem $B_\nu(T(\tau_\nu))$ is drawn dashed.

Radiative transfer: Solar limb darkening observations

Observed limb darkening on the Sun – the solar disk.

Observed limb darkening on the Sun – intensity vs. $\mu = \cos \theta$ for different wavelengths from (Stix, 1989).

Radiative transfer: Molecular absorption

At GHz and THz frequencies we observe in zenith direction if we assume optically thin conditions ($\tau_\nu \ll 1$) and neglect the galactic background

$$I_\nu = \int_0^\infty B_\nu(T(z)) \kappa_\nu(z) dz$$

For a line at center frequency $\nu_{nm} = (E_m - E_n)/h$ for a transition from state $n \rightarrow m$ of a molecule X

$$\kappa_\nu \propto \underbrace{n_{X,n}}_{\substack{\text{density of} \\ \text{X in state } n}} \nu \underbrace{\Psi(\nu - \nu_{nm})}_{\text{line shape}} \underbrace{\left(1 - e^{-\frac{E_m - E_n}{k_B T}}\right)}_{\text{induced emission}}$$

- The density $n_{X,n}$ is related to the concentration c_X

$$n_{X,n} = n_{\text{air}} c_X \frac{g_n e^{-\frac{E_n}{k_B T}}}{Z(T)}$$

with g_n the degeneracy of state n and $Z(T)$ is the partition function.

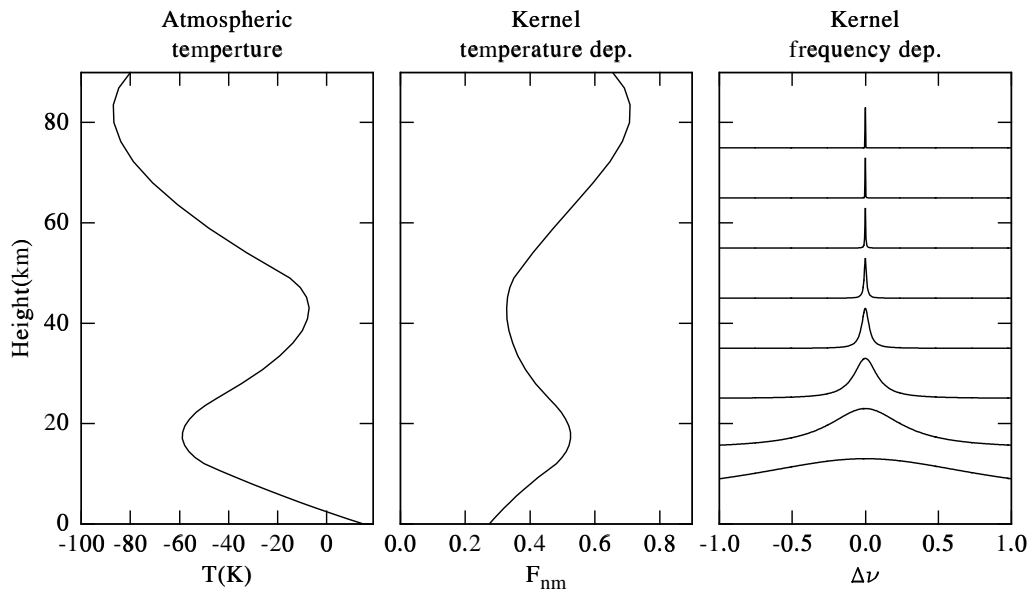
- The line shape collision dominated and well modelled by a Lorentzian line profile

$$\Psi(\nu - \nu_{nm}) = \frac{\Delta\nu_C}{(\nu - \nu_{nm})^2 + (\Delta\nu_C)^2}$$

with width

$$\Delta\nu_C \simeq \Delta\nu_0 \frac{p}{p_0}$$

Radiative transfer: Trace gas inversion



Typical temperature profile of the Earth's atmosphere, factor $F_{n,m}$ and line shape of kernel at various heights.

Insertion yields the inversion problem

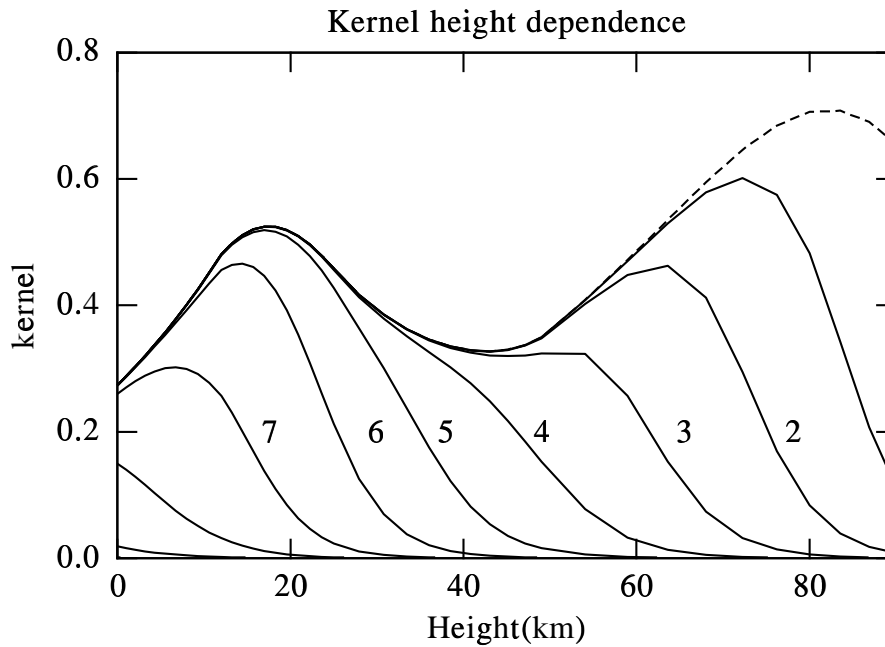
$$\underbrace{I_\nu}_{\text{data}} = \int_0^\infty \underbrace{F_{nm}(T(z))}_{\text{model 1}} \underbrace{\frac{\left(\frac{p(z)}{p_0}\right)^2}{\left(\frac{\nu - \nu_{nm}}{\Delta\nu_0}\right)^2 + \left(\frac{p(z)}{p_0}\right)^2}}_{\text{kernel}} \underbrace{c_X(z)}_{\text{model 2}} dz$$

where the T dependence is concentrated in

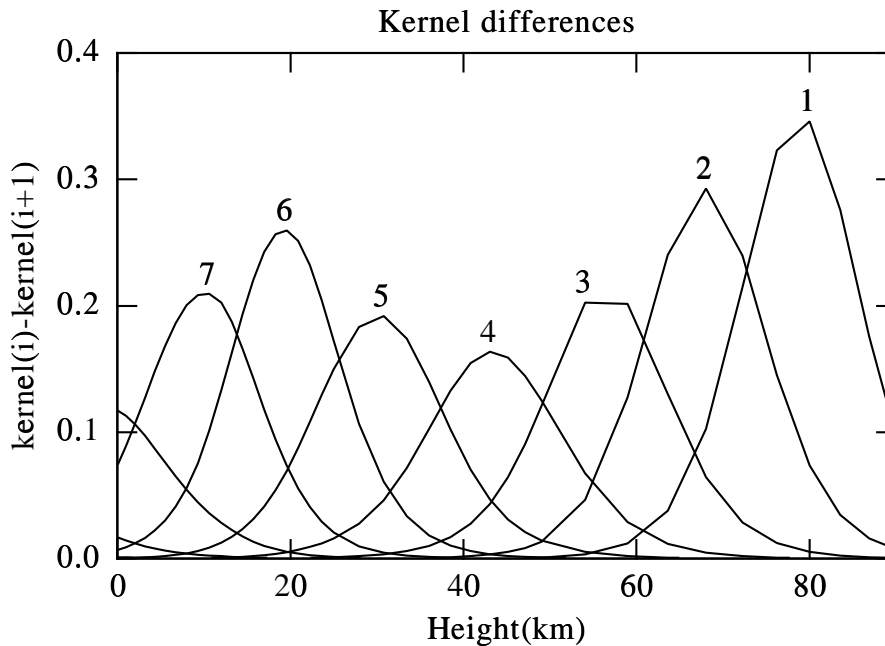
$$F_{nm}(T) \propto B_{\nu_{nm}}(T) \frac{E_m - E_n}{k_B T} \frac{e^{-\frac{E_n}{k_B T}} - e^{-\frac{E_m}{k_B T}}}{Z(T)}$$

- $X = \text{CO}_2$ or O are well mixed so that $c_X = \text{const}$
 \rightarrow solve for $F_{nm}(T(z))$, i.e. $T(z)$.
- If $T(z)$ is known, solve for $c_X(z)$ of more exotic trace gases.

Radiative transfer: Trace gas inversion kernel



Height dependence of kernel functions with increasing frequency offset.



Height dependence of difference between neighbouring kernel functions.

- Combinations of the inversion equation for different $\nu - \nu_{nm}$ may give better kernels.

Helioseismology: Fundamental properties

We assume hydrostatic equilibrium

$$\nabla p_0 = \mathbf{g}\rho_0 \quad \text{and} \quad \nabla \cdot \mathbf{g} = -4\pi G\rho_0$$

where in our notation $g = \hat{\mathbf{r}} \cdot \mathbf{g}$ is negative.

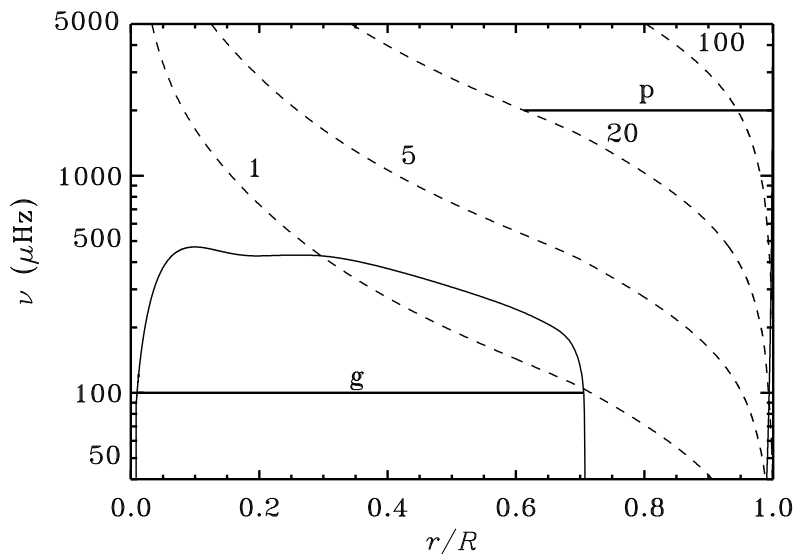
The propagation of waves in the Sun is controlled by three further parameters

$$\text{Acoustic speed: } c_s^2 = \frac{\gamma p_0}{\rho_0}$$

$$\text{Brunt-Vaisälä frequency: } N^2 = |g| \left(\frac{\partial_r p_0}{\gamma p_0} - \frac{\partial_r \rho_0}{\rho_0} \right)$$

$$= \frac{|g|}{\gamma} \left(\frac{p_0}{\rho_0^\gamma} \right)^{-1} \frac{\partial}{\partial r} \left(\frac{p_0}{\rho_0^\gamma} \right)$$

$$\text{atmospheric scale height } H = \frac{p_0}{|g|\rho_0} = \frac{c_s^2}{\gamma|g|}$$



Variation of N (solid) and $c_s k_h$ (dashed) with distance from the center of the Sun for $k_h \simeq \sqrt{l(l+1)}/r$ (Christensen-Daalsgaard, 1998).

Helioseismology: Lagrangian perturbations

The inversion equation is derived from a variational principle which involves the integration of perturbations over the whole solar volume.

→ it is advantageous to change from Eulerian variables

$$\frac{d}{dt}\mathbf{v} = -\frac{1}{\rho}\nabla p + \mathbf{g}$$

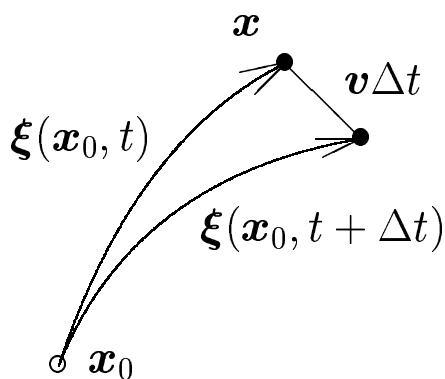
$$\frac{d}{dt}\rho = -\rho\nabla\cdot\mathbf{v} \quad \text{and} \quad \frac{d}{dt}p = -\gamma p\nabla\cdot\mathbf{v}$$

to Lagrangian variables:

$$\ddot{\boldsymbol{\xi}} = (\gamma - 1)\mathbf{g}(\nabla\cdot\boldsymbol{\xi}) + c_s^2\nabla(\nabla\cdot\boldsymbol{\xi}) + \nabla(\mathbf{g}\cdot\boldsymbol{\xi}) \equiv -\mathcal{A}(\boldsymbol{\xi})$$

$$\delta_L\rho = -\rho_0(\nabla\cdot\boldsymbol{\xi}) \quad \text{and} \quad \delta_L p = -\gamma p_0(\nabla\cdot\boldsymbol{\xi})$$

On the Sun's surface: $p_0 = \delta_L p = 0$



Relation between Eulerian and Lagrangian perturbations for fixed t :

$$\mathbf{x} = \mathbf{x}_0 + \boldsymbol{\xi}(\mathbf{x}_0, t)$$

$$\mathbf{v}(\mathbf{x}, t) = \dot{\boldsymbol{\xi}}(\mathbf{x}_0, t)$$

- The velocity perturbations on the Sun's surface $\dot{\boldsymbol{\xi}}$ can be observed. Usually their FT are considered (for plane parallel geometry):

$$\boldsymbol{\xi}(\mathbf{x}_0, t) = \sum_{\mathbf{k}_h, \omega} \boldsymbol{\xi}_{\mathbf{k}_h, \omega}(z_0) e^{i(\mathbf{k}_h \mathbf{x}_0 - \omega t)} + c.c.$$

Helioseismology: Short wavelength approximation

To qualitatively understand the observations we simplify:

- plane parallel geometry ($r \rightarrow z$)
- no gravity waves (g-modes are yet undetected)

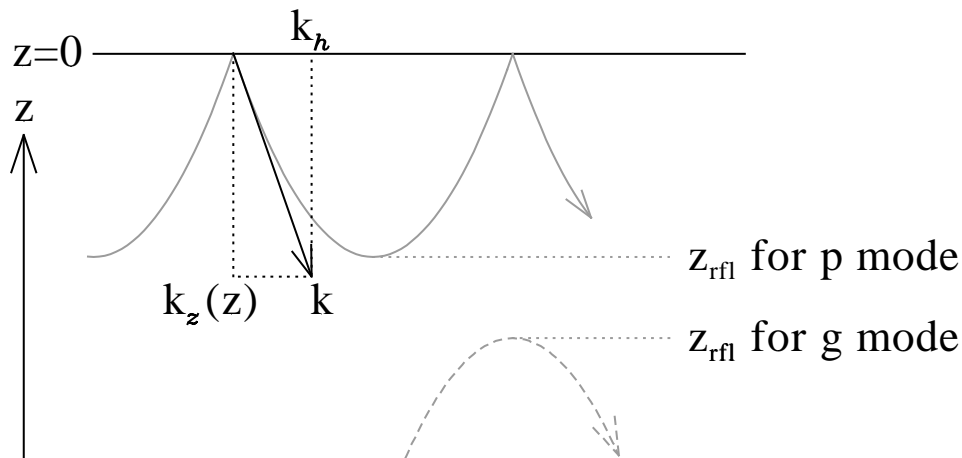
For $Hk_z \gg 1$ the Lagrangian momentum equation yields

$$\text{div} \ddot{\xi} - c_s^2 \Delta \text{div} \xi \simeq 0$$

For an observed mode with horizontal wavenumber k_h and frequency ω fixed

$$k_z(z)^2 \simeq \frac{\omega^2}{c_s^2(z)} - k_h^2$$

Since c_s increases with depth we have reflection between the surface $z = 0$ and some z_{rfl} inside the Sun.



Propagation paths of an acoustic wave (p-mode) in the Sun. In comparison, the propagation of gravity waves (g-mode) is dashed.

- In between the reflection points, the wave must have an integer number of nodes

$$\int_{z_{\text{rfl}}}^0 k_z(z) dz = (n + \alpha_{\text{rfl}} + \alpha_0)\pi$$

where $\alpha_{0,\text{rfl}}$ are phase corrections at the respective reflection height (from observations: $\alpha_{\text{rfl}} + \alpha_0 \simeq 1.45$)

Helioseismology: Short wavelength dispersion

With $N^2 \simeq 0$, we can roughly estimate how c_s increases with depth inside the convection zone:

$$\begin{aligned} \frac{1}{c_s^2} \frac{dc_s^2}{dz} &= \left(\frac{1}{p_0} \frac{dp_0}{dz} - \frac{1}{\rho_0} \frac{d\rho_0}{dz} \right) \\ &= \left(\frac{1}{p_0} \frac{dp_0}{dz} - \frac{1}{\gamma p_0} \frac{dp_0}{dz} \right) = \left(1 - \frac{1}{\gamma} \right) \frac{g\rho_0}{p_0} = (\gamma - 1) \frac{g}{c_s^2} \end{aligned}$$

Insert $c_s^2(z)$ into the equation for $k_z^2(z)$

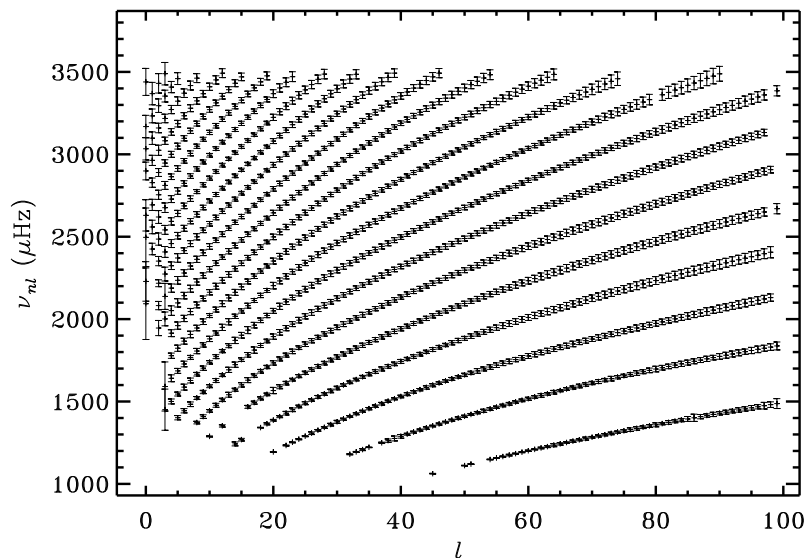
$$k_z^2(z) \simeq \frac{\omega^2}{(\gamma - 1)gz} - k_h^2$$

Insertion of $k_z(z)$ into node number integral gives

$$\int_{z_{\text{rfl}}}^0 \sqrt{\frac{\omega^2}{(\gamma - 1)gz} - k_h^2} dz = k_h |z_{\text{rfl}}| \frac{\pi}{2} = (n + \alpha_{\text{rfl}} + \alpha_0)\pi$$

where $z_{\text{rfl}} = (\omega/k_h)^2 / (\gamma - 1)g$. The last equation yields the observable relation between the frequency and horizontal wavenumber

$$\omega^2 = \omega_{\mathbf{k}_h, n}^2 \simeq 2(\gamma - 1)g (n + \alpha_{\text{rfl}} + \alpha_0)k_h$$



Observed dispersion for acoustic modes inside the Sun with $l \simeq k_h R_\odot$

Helioseismology: The variational principle

Introduction of the Fourier transform in the Lagrangian momentum equation gives

$$\omega_{\mathbf{k}_h, n}^2 \boldsymbol{\xi}_{\mathbf{k}_h, n} = \mathcal{A}_{\mathbf{k}_h}(\boldsymbol{\xi}_{\mathbf{k}_h, n}) \quad (1)$$

an eigenvalue problem for eigenvalues $\omega_{\mathbf{k}_h, n}^2$ and eigenstates $\boldsymbol{\xi}_{\mathbf{k}_h, n}$.

Since $\mathcal{A}_{\mathbf{k}_h}$ is hermitian, the eigenstates span a Hilbert space with normalization

$$\int_{-\infty}^0 \rho_0 (\boldsymbol{\xi}_{\mathbf{k}_h, m}^*(z) \cdot \boldsymbol{\xi}_{\mathbf{k}_h, n}(z)) dz = \delta_{m, n} M_{\mathbf{k}_h, n} \quad (\text{mode inertia}) \quad (2)$$

hence the above eigenvalue equation (1) can also be written as

$$\omega_{\mathbf{k}_h, n}^2 M_{\mathbf{k}_h, n} = \int_{-\infty}^0 \rho_0 (\boldsymbol{\xi}_{\mathbf{k}_h, n}^* \cdot \mathcal{A}_{\mathbf{k}_h}(\boldsymbol{\xi}_{\mathbf{k}_h, n})) dz \quad (3)$$

• If the calculated $\omega_{\mathbf{k}_h, n}^2$ do not agree with the observed frequencies, we have to vary ρ_0 (and all other parameters accordingly) in our model:

$$\begin{aligned} \rho_0 \rightarrow \rho_0 + \delta\rho_0 \quad \text{causes} \quad p_0 \rightarrow p_0 + \delta p_0 \quad ; \quad g \rightarrow g + \delta g \\ \mathcal{A}_{\mathbf{k}_h} \rightarrow \mathcal{A}_{\mathbf{k}_h} + \delta\mathcal{A}_{\mathbf{k}_h} \quad ; \quad \boldsymbol{\xi}_{\mathbf{k}_h, n} \rightarrow \boldsymbol{\xi}_{\mathbf{k}_h, n} + \delta\boldsymbol{\xi}_{\mathbf{k}_h, n} \\ \text{and finally} \quad \omega_{\mathbf{k}_h, n}^2 \rightarrow \omega_{\mathbf{k}_h, n}^2 + \delta(\omega_{\mathbf{k}_h, n}^2) \end{aligned}$$

We may vary the eigenvalue equation (1) but then we need the perturbations of the eigenstates as well. A more convenient approach is to vary (3). The procedure then is almost identical to conventional perturbation theory in quantum mechanics. In any case, the orthogonality (2) and the mode inertia $M_{\mathbf{k}_h, n}$ remain invariant under the variation.

Helioseismology: The inversion problem

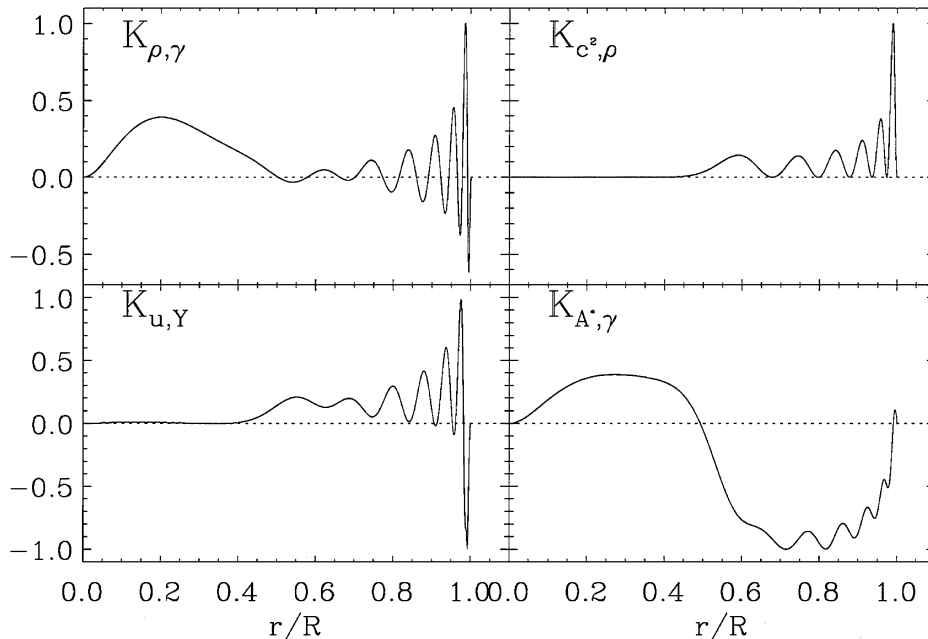
The variation of (3) yields

$$\delta(\omega_{\mathbf{k}_h,n}^2)M_{\mathbf{k}_h,n} = \int_{-\infty}^0 \rho_0 (\boldsymbol{\xi}_{\mathbf{k}_h,n}^* \cdot \delta \mathcal{A}_{\mathbf{k}_h}(\boldsymbol{\xi}_{\mathbf{k}_h,n})) dz$$

Hence the variation of the eigenvalue is obtained from the variation of the operator $\mathcal{A}_{\mathbf{k}_h}$ with respect to the unperturbed eigenfunctions. The perturbed eigenfunctions are not required.

Next, relate the variation $\delta \mathcal{A}_{\mathbf{k}_h}$ to the appropriate variation $\delta \rho_0$ (Fréchet derivative $\propto \mathcal{K}_{\mathbf{k}_k}$). We finally obtain after renormalization:

$$\underbrace{\frac{\delta(\omega_{\mathbf{k}_h,n}^2)}{\omega_{\mathbf{k}_h,n}^2}}_{\text{data}} = \int_{-\infty}^0 \underbrace{\mathcal{K}_{\mathbf{k}_h}(\boldsymbol{\xi}_{\mathbf{k}_h,n}^*, \boldsymbol{\xi}_{\mathbf{k}_h,n})}_{\text{kernel}} \underbrace{\frac{\delta \rho_0}{\rho_0}}_{\text{model}} dz$$



Kernel functions $K_{\mathbf{k}_h}$ for an acoustic mode $k_h \simeq \sqrt{l(l+1)}/R_\odot$ with $l = 10$ and eigenfunction order $n = 6$. The first subscript parameter is varied, the second fixed. Here, $u = c_s^2/\gamma$, $A^* \propto N^2/|g|$ and $Y \propto \text{He abundance}$ (Kosovichev, 1999).