HYDRODYNAMICS

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Bulletin of exercises n° 2: Constitutive relations for a linearly viscous fluid.

In order to obtain the form of the stress tensor for *linearly viscous fluids* the following assumptions are made:

(i) The stress tensor at any point $\mathbf{x}$ of the medium depends on the thermodynamic state and on the gradient-of-velocity tensor at this point.

(ii) When the fluid is at rest (or there are no gradients of velocity), the constitutive relation for the stress tensor must reduce to that of hydrostatics.

From (i) and (ii) it follows that the constitutive relation for σ_{ij} must be of the form

$$\sigma_{ij} = -p\delta_{ij} + \tau_{ij}(v_{r,s}, \rho, T)\,, \qquad \text{with} \qquad \tau_{ij}(0, \rho, T) = 0\,, \tag{1}$$

where we have chosen the variables ρ, T to represent the thermodynamic state of the fluid. The coefficient p which multiplies δ_{ij} is the *thermodynamic* pressure, which coincides with the *mechanical pressure* when the fluid is at rest (i.e., with the *hydrostatic* pressure).

In this excercise the form of the nine functions τ_{ij} is deduced.

1. It is required that the relationship between $\hat{\boldsymbol{\sigma}}$ and $\nabla\mathbf{v}$ be linear (this is the simplest extension of the constitutive relation $\hat{\boldsymbol{\sigma}} = -p\hat{\boldsymbol{I}}$ which includes the dependence of $\hat{\boldsymbol{\sigma}}$ with $\nabla\mathbf{v}$ and is compatible with "Navier's hypothesis" for viscosity).

Show that this linear relationship along with the symmetry of $\hat{\boldsymbol{\sigma}}$ imply that (1) can be written in the form

$$\sigma_{ij} = -p\delta_{ij} + C_{ijrs}v_{r,s}\,, \qquad \text{with} \qquad C_{ijrs} = C_{jirs}\,, \tag{2}$$

where $\hat{\boldsymbol{C}} = (C_{ijrs})$ is a fourth-order tensor whose components are functions of ρ and T. How many components has $\hat{\boldsymbol{C}}$?

Newtonian fluids

2. Prove that if, further, isotropy is assumed (i.e., that there are no preferred directions in space), the number of independent elements of the tensor $\hat{\boldsymbol{C}}$ reduces to only two. Write the general form of C_{ijrs}.

Hint: The most general isotropic fourth-order tensor has the form

$$A_{ijrs} = \lambda\delta_{ij}\delta_{rs} + \mu(\delta_{ir}\delta_{js} + \delta_{is}\delta_{jr}) + \eta(\delta_{ir}\delta_{js} - \delta_{is}\delta_{jr})\,. \tag{3}$$

3. Prove that $C_{ijrs}v_{r,s} = C_{ijrs}D_{rs}$, where $\hat{\boldsymbol{D}}$ is the symmetric part of $\nabla\mathbf{v}$; that is, the antisymmetric part of $\nabla\mathbf{v}$ is not involved in the constitutive relations. Discuss the physical interpretation of this fact.

4. Write the resulting constitutive relations (which describe the model called "Newtonian fluid"). Obtain the relation existing between the thermodynamic and the mechanical pressures.

[NOTE: The mechanical pressure at $\mathbf{x}$ is defined as minus the mean value of the normal stresses at $\mathbf{x}$].